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High Reynolds number flow past a flat plate with strong blowing

By J. B. KLEMP† AND A. ACRIVOS
Department of Chemical Engineering, Stanford University

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For the uniform flow past a semi-infinite flat plate subject to a blowing velocity profile equal to $C(U\nu/x)^{\frac{1}{2}}$ the conventional boundary-layer approximations break down as C approaches 0·6192. Here, we consider the structure of the flow for large Reynolds numbers R when C exceeds this critical value. It is shown that, for $C > 0{\cdot}6192$, a region containing injected fluid $O(R^{-\frac{1}{3}})$ in thickness forms directly above the plate. To a first approximation the flow in this region is inviscid and the pressure a function of x only. This blowing region is separated from the free stream by a free shear boundary layer of thickness $O(R^{-\frac{1}{2}})$. Thus the flow domain consists of three distinct regions which interact to yield a similarity solution valid for large values of Rx. This solution is then extended to higher order by expanding the stream function in each region in powers of $(Rx)^{-\frac{1}{6}}$ and evaluating the first four terms in the resulting series using standard matching techniques. Finally, more general blowing profiles which also lead to boundary-layer 'blow off' are considered and an expression, valid far downstream of boundary-layer detachment, is derived for the position of the streamline separating the injected fluid from that of the free stream. For the case of uniform blowing the blowing region takes on the shape of a wedge, indicating that no solution can exist for the corresponding external flow if the plate is truly semi-infinite.

1. Introduction

It has been known for quite a while, in fact since the early days of Prandtl, that suction or blowing can have a profound effect on the structure of the laminar boundary layer adjacent to a porous surface. For example, it has been shown that a small amount of suction can retard or even prevent boundary-layer separation and that a small amount of blowing, as in transpiration cooling, can substantially decrease the rate of heat transfer from a hot external fluid to a cold surface. This subject has therefore been studied in some detail by a number of investigators.

In the case of blowing, with which we shall be concerned here, the nature of the flow strongly depends on whether v_0, the blowing velocity relative to that of the free stream, is $O(R^{-\frac{1}{2}})$ or $O(1)$, R denoting the Reynolds number. Pretsch (1944) was one of the first to deal with a problem in the former category for favourable pressure gradients by considering the laminar boundary-layer flow

† Present address: National Center for Atmospheric Research, Boulder, Colorado.

past a wedge with v_0 proportional to $x^{\frac{1}{2}(1-m)}$ so that a similarity transformation leading to the familiar Falkner–Skan equation could be applied. Pretsch, who later extended his analysis to more general surfaces and blowing velocity profiles, found, for a favourable pressure gradient and in the limit of strong $O(R^{-\frac{1}{2}})$ blowing, that the viscous term in the boundary-layer equations could be neglected in the region near the surface and the surface shear stress could be computed from the solution of the inviscid equations alone. The same problem was later studied by Acrivos (1962) using the method of inner and outer expansions. Acrivos showed, again in the limit of strong $O(R^{-\frac{1}{2}})$ blowing, that the boundary layer can be divided into two overlapping regions: a relatively thick inner region containing most of the fluid emitted from the surface in which, as was done by Pretsch, viscosity can be neglected to a first approximation, and a thin viscous region of the normal boundary-layer scale where the fluid velocity is close to its value as given by potential flow theory. An interesting feature of this flow is that, for a constant property fluid, higher order approximations to the inner solution, and therefore to the surface shear stress, can be constructed using a regular perturbation expansion in terms of a suitably chosen blowing parameter without having to take into account the existence of the viscous region. Another point worth noting is that this viscous layer, which plays an important role in determining the rate of mass transfer from an ablating surface to the external flow (Acrivos 1962), refers to the mixing zone between the injected fluid and that of the main stream, and therefore, to the layer where the bulk of the vorticity transfer occurs between the inner rotational region and the external potential flow. A detailed analysis of this problem along the lines summarized above was given recently by Watson (1966). Elliott (1968) extended the theory to the general two-dimensional case.

The general features of the flow just described are also depicted in figure 1(*a*) for the case of a favourable pressure gradient and in the limit of strong $O(R^{-\frac{1}{2}})$ blowing. We have denoted the inner, or blowing region as I, the vorticity boundary layer as II, and the irrotational region as III. Region I is of course much thicker than II, however both are $O(R^{-\frac{1}{2}})$ in thickness and therefore much thinner than any characteristic dimension of the body, provided the Reynolds number R is sufficiently large. Consequently, the potential flow remains unaffected by the blowing for $R \gg 1$, and hence the pressure term dp/dx appearing in the boundary-layer equations is the same whether there is blowing or not.

The flow pattern changes drastically, however, when the blowing velocity is of the same order of magnitude as the speed of the free stream, i.e. $v_0 \sim O(1)$. As shown by Cole & Aroesti (1968) and illustrated in figure 1(*b*) the thickness of the blowing region relative to some characteristic dimension of the body is now $O(1)$, hence the pressure along II is no longer independent of the blowing rate but must be determined as part of the solution. Nevertheless the problem does retain one of the simplifying features of the case discussed earlier in that the solution in regions I and III can again be obtained on the basis of the inviscid equations, i.e. without reference to the presence of the vorticity boundary layer II. A general procedure for constructing such solutions whenever the thickness of the blowing region I, although independent of R, is still numerically small compared to a typical dimension of the body was given by Cole & Aroesti (1968). Wallace & Kemp (1969) later relaxed this restriction. Extensions of the theory to supersonic and hypersonic speeds have recently been described by Lees & Chapkis (1969) and Thomas (1969) whose papers also refer to most of the earlier theoretical and experimental studies on the subject.

In what follows we shall examine yet another blowing problem whose solution, as will be seen, corresponds neither to figure 1 (*a*) nor to figure 1 (*b*). This is the classical case of laminar boundary-layer flow past a flat plate with a blowing velocity $v_0 = C/(Rx)^{\frac{1}{2}}$, C being a constant, where, in the absence of blowing, the pressure gradient is identically zero.

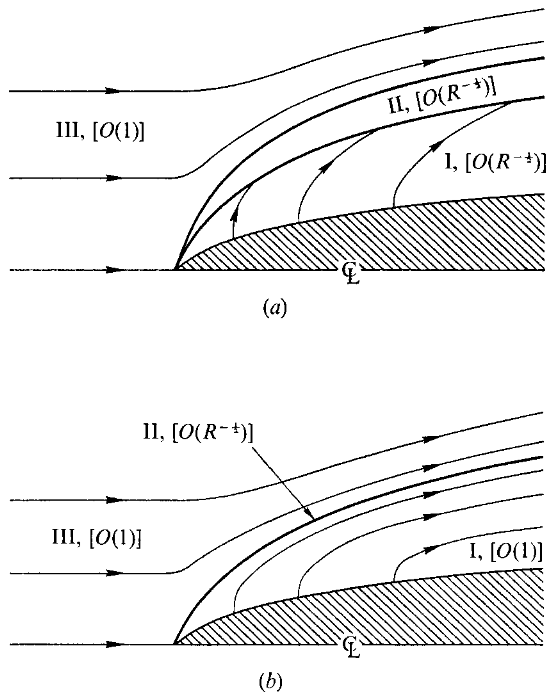

FIGURE 1. Structure of the flow for favourable pressure gradients (for both (*a*) and (*b*) the thickness of the body is $O(1)$ while the thickness of each region is indicated in brackets). (*a*) Strong blowing, $v_0 = O(R^{-\frac{1}{2}})$. (*b*) Massive blowing, $v_0 = O(1)$.

It is well known (Rosenhead 1963, p. 242) that the problem thus stated can be solved by the familiar Blasius similarity transformation provided that $C < 0{\cdot}6192$. However, as C approaches this critical value from below, the surface shear stress is found to decrease rapidly towards zero and the displacement thickness to increase without limit thus implying that at $C = 0{\cdot}6192$ the boundary layer is 'blown off'. The form of the solution as $C \to 0{\cdot}6192$ was obtained analytically by Kassoy (1970) and was found to be in good agreement with the extensive numerical calculations by Emmons & Leigh (1954).

The purpose of this paper is to consider the structure of the flow when C exceeds the above mentioned critical value, a problem also studied recently by Kassoy (1971), whose work, however, became available to us only after the completion of the present analysis. As will be seen shortly the appropriate solution (for $C > 0{\cdot}6192$), although retaining some of the features shown in figure 1, has a number of distinctive properties all of its own. Specifically, the blowing region I is neither $O(R^{-\frac{1}{2}})$ nor $O(1)$ in thickness but rather $O(R^{-\frac{1}{2}})$; region II, although still $O(R^{-\frac{1}{2}})$ thick, now becomes a velocity rather than a vorticity boundary layer in the sense that there is an $O(1)$ change in velocity across it; the external irrotational stream imparts an $O(R^{-\frac{1}{3}})$ favourable pressure gradient to regions I and II which depends on the particular value of the blowing parameter C_j and, finally, in contrast to the two cases discussed earlier, the solutions in the three regions are now interconnected in that it is no longer possible to obtain the flow in I and III without having to take into account the existence of II. Apart from being of interest in its own right, the solution to be developed below is believed to be of more general significance in that it refers to a rather unique example of a post-separation flow which can be successfully treated theoretically using boundary-layer techniques plus asymptotic expansions.

In the next section we present the basic properties of this solution and in particular its asymptotic form as $Rx \to \infty$. The latter has already been derived by Kassoy (1971). Nevertheless, we shall develop this asymptotic solution in some detail, not only because our method of solution differs somewhat from Kassoy's, but because it will pave the way for the more complete analysis that follows. Thus, using this asymptotic solution as a starting point, we increase the range of its validity in § 3 by constructing the first four terms of an asymptotic expansion in terms of the small parameters $(Rx)^{-\frac{1}{6}}$. To complete the picture, we briefly discuss in § 4 the effects of more general blowing velocity profiles $v_0(x)$.

2. Principal properties of the solution and its asymptotic form for large Rx

We consider the laminar flow of an incompressible constant property fluid past a semi-infinite horizontal flat plate. The variables are rendered dimensionless in the usual manner using U, the speed of the free stream, as the characteristic velocity and some arbitrary length, l, as the characteristic dimension. The Reynolds number R is defined as Ul/ν, ν being the kinematic viscosity. The system then consists of the familiar dimensionless Navier–Stokes and continuity equations expressed in the standard Cartesian co-ordinates, plus the following boundary conditions:

$$\left.\begin{array}{lll} u = 1, & v = 0 & \text{at} \quad x = -\infty, \\ \partial u/\partial y = 0, & v = 0 & \text{at} \quad y = 0, \quad x < 0, \\ u = 0, & v = C/(Rx)^{\frac{1}{2}} & \text{at} \quad y = 0, \quad x > 0, \quad \text{with} \quad C > 0{\cdot}6192. \end{array}\right\} \tag{2.1}$$

Evidently, the solution to the above problem should be independent of the choice of l.

In seeking to construct a solution that applies at large R we are guided by the fact, already noted in the introduction, that the boundary layer which normally would be expected to lie on the plate is blown off as C is increased to the critical value $0{\cdot}6192$. Thus for $C > 0{\cdot}6192$ the blowing region must be thicker than $O(R^{-\frac{1}{2}})$, which in the limit of large Reynolds number means that it will be inviscid to a first approximation. Consequently the flow pattern here is in one respect qualitatively similar to that of the two cases considered earlier and depicted in figure 1, in the sense that viscous forces are small in comparison to the inertia terms everywhere except in a thin layer separating the blowing region from the external irrotational stream. Let us suppose then that the blowing region occupies the space $0 \leqslant y \leqslant F(R)\,h(x)$, $x > 0$, where $h(x)$ is as yet unknown and $F(R) > O(R^{-\frac{1}{2}})$. We further suppose, subject to *a posteriori* verification, that $F(R) \to 0$ as $R \to \infty$. Also, because of the boundary condition at $y = 0$, $v = O(R^{-\frac{1}{2}})$ throughout the blowing region and, by continuity,

$$u = O(F^{-1}R^{-\frac{1}{2}}),$$

which, to a first approximation, vanishes as $R \to \infty$. Therefore by Bernoulli's equation, the pressure in the blowing region is $O(F^{-2}R^{-1})$ plus an additive constant which may be neglected. In addition, from the y component of the Navier–Stokes equation, the pressure drop across I is $O(R^{-1})$ and that across II is $O(R^{-\frac{1}{2}}F)$, consequently, the pressure in the blowing region must equal the pressure at the lower edge of III as given by the irrotational flow solution. From thin air foil theory, however, the latter is $O(F)$, hence $F(R) = R^{-\frac{1}{3}}$.

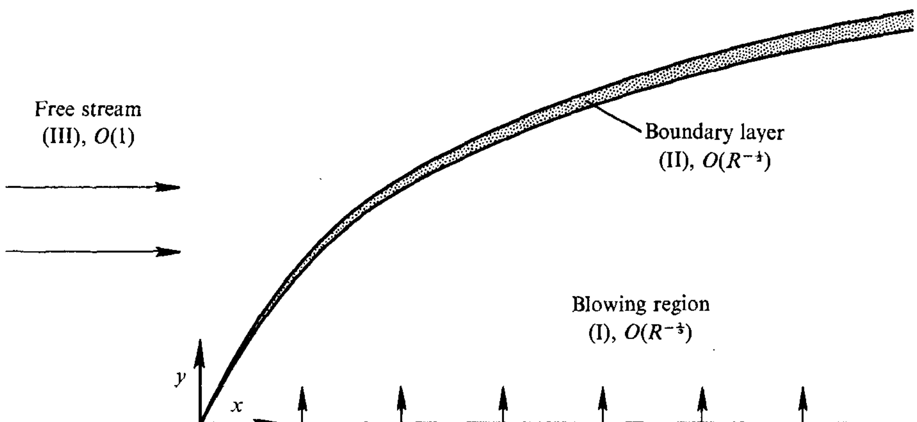

FIGURE 2. Uniform flow past a flat plate with $v_0(x) = C(Rx)^{-\frac{1}{2}}$, $C > 0{\cdot}6192$.

The basic features of the flow when $C > 0{\cdot}6192$ and $R \gg 1$ are then as sketched in figure 2. In the blowing region I viscous forces are negligible to a first approximation, the scale of the lateral dimension y is $O(R^{-\frac{1}{3}})$, and u and v are $O(R^{-\frac{1}{6}})$ and $O(R^{-\frac{1}{2}})$ respectively. Region II is a conventional boundary layer separating an effectively stagnant fluid from an essentially uniform stream whose speed is $O(1)$. Finally, region III refers to the irrotational flow past a body whose thickness ratio is $O(R^{-\frac{1}{3}})$.

The flow in region II can be immediately ascertained from Lock's (1951) classical study of the velocity distribution in the laminar boundary layer between parallel streams. Denoting by $\Psi = 0$ the streamline separating the injected fluid from that of the free stream, we have from Lock's (1951) solution that:

$$\Psi_{\mathrm{II}}(x, -\infty) = -2\beta_1 (x/R)^{\frac{1}{2}},$$

where $\Psi_{II}(x, -\infty)$ refers to the value of the stream function at the lower edge of the boundary-layer region II and $\beta_1 = 0{\cdot}6192$. Consequently the boundary conditions for the solution inside I, the blowing region, are:

$$\left.\begin{aligned} \Psi_I &= -2C(x/R)^{\frac{1}{2}}, \quad \partial\Psi_I/\partial y = 0 \quad \text{at} \quad y = 0, \\ \Psi_I &= -2\beta_1(x/R)^{\frac{1}{2}} \quad \text{at} \quad y = R^{-\frac{1}{3}}h(x), \text{ the upper boundary of I.} \end{aligned}\right\} \tag{2.2}$$

Noting that β_1 equals exactly the critical value of C at which the conventional Blasius type boundary-layer solution 'blows off', we see that the flow pattern as depicted in figure 2, with the thickness of the blowing region increasing downstream, is possible only as long as the rate of fluid entrainment from I into II is less than the rate at which fluid is being injected into I along the porous surface $y = 0$; this means, of course, that C must exceed $0{\cdot}6192$. Otherwise, i.e. for $C < 0{\cdot}6192$, the boundary layer is entirely capable of entraining all the injected fluid and blow off cannot occur.

The form of Ψ_I along the boundaries of I and the absence of a natural characteristic length l for the problem suggest now the existence of a similarity solution which would be expected to apply for $R \to \infty$; in fact, as will be shown shortly, it holds for $(Rx)^{\frac{1}{6}} \gg 1$. We suppose therefore that the boundary between I and II is given by $y = \alpha_1 R^{-\frac{1}{3}} x^n$ with α_1 being some dimensionless positive constant to be determined. However, since the solution must be independent of the value of the arbitrary characteristic length l which enters into y, x and R, the only possible choice of n is $\frac{2}{3}$, i.e. the blowing region I extends from $y = 0$ to

$$y = R^{-\frac{1}{3}}h(x) = \alpha_1 R^{-\frac{1}{3}} x^{\frac{2}{3}}. \tag{2.3}$$

Consequently, from thin airfoil theory (Van Dyke 1964), the pressure that is impressed on I and II by the potential flow in III is

$$P_{III}(x,0) = P_{III\infty} - \frac{2}{3\pi} R^{-\frac{1}{3}} \alpha_1 \int_0^\infty \frac{z^{-\frac{1}{3}}}{x-z}\,dz = P_{III\infty} + \frac{2}{3^{\frac{3}{2}}}\alpha_1 (Rx)^{-\frac{1}{3}}, \tag{2.4}$$

where $P_{III\infty}$, to be set equal to zero henceforth, denotes the pressure of the uniform stream, and the integral refers to its Cauchy principal value. The basic equation within I is then

$$\frac{\partial\Psi}{\partial y}\frac{\partial^2\Psi}{\partial x\,\partial y} - \frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2} = \frac{2\alpha_1}{3^{\frac{5}{2}}} R^{-\frac{1}{3}} x^{-\frac{4}{3}}, \tag{2.5}$$

with boundary conditions given by (2.2) and (2.3).

The substitution

$$\Psi = -2CR^{-\frac{1}{2}}x^{\frac{1}{2}}g(\zeta), \quad \zeta \equiv R^{\frac{1}{3}} y x^{-\frac{2}{3}} \tag{2.6}$$

reduces (2.5) to

$$g g'' + \tfrac{1}{3}(g')^2 + \alpha_1/3^{\frac{5}{2}}C^2 = 0, \tag{2.7}$$

with boundary conditions

$$g(0) = 1, \quad g'(0) = 0, \quad g(\alpha_1) = \beta_1/C.$$

The solution satisfying the first two of the above is

$$\zeta = C\alpha_1^{-\frac{1}{2}} 3^{\frac{3}{4}} \int_g^1 \frac{d\tau}{(\tau^{-\frac{2}{3}} - 1)^{\frac{1}{2}}} = C\alpha_1^{-\frac{1}{2}} 3^{\frac{3}{4}} (2 + g^{\frac{2}{3}})(1 - g^{\frac{2}{3}})^{\frac{1}{2}}, \tag{2.8}$$

hence, on account of the third condition,

$$\alpha_1 = \beta_1^{\frac{2}{3}}\, 3^{\frac{1}{2}} \lambda^{-1}(2+\lambda)^{\frac{2}{3}}(1-\lambda)^{\frac{1}{3}}, \quad \text{with} \quad \lambda \equiv (\beta_1/C)^{\frac{2}{3}}. \tag{2.9}$$

The above together with (2.6) and (2.8) completes then the similarity solution which, of course, agrees with that given by Kassoy (1971). The expression for α_1 behaves as expected, for α_1 approaches zero or infinity according to whether $C \to 0{\cdot}6192$ or ∞. Also, the flow pattern in the blowing region, such as that depicted in figure 3 for $\lambda = 0{\cdot}63$ (corresponding to $C = 2\beta_1$), conforms with the general requirements of the solution as discussed earlier and in connexion with figure 2. In particular the solution clearly underlines the strong coupling that exists among all the three regions in this problem, in contrast to the other blowing problems mentioned in the introduction. Thus, the fact that the thickness of I is $O(R^{-\frac{1}{3}})$ – a result arrived at by matching the pressure across II – and hence, by continuity, that u is $O(R^{-\frac{1}{6}})$ within I, allowed us to solve for the

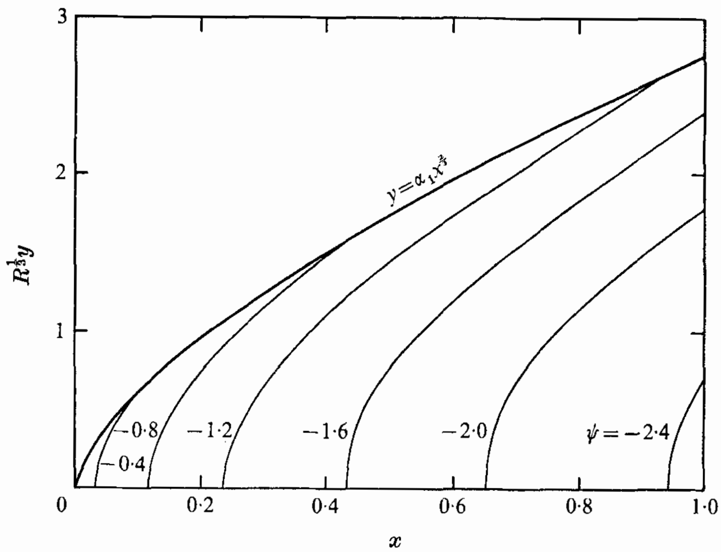

FIGURE 3. Streamline pattern in blowing region for $\lambda = 0{\cdot}63$ ($C = 2\beta_1$).

flow inside the boundary layer by neglecting the velocity in the blowing region I. In turn this provided us with one of the boundary conditions in (2.2) for the solution in I. Finally, the coupling between I and III was effected through the pressure gradient in (2.5) which came from III via (2.4). However, the latter contained the coefficient α_1 which was determined only after finding the complete solution within the blowing region and applying the matching condition in the overlap domain between I and II. This coupling among the solutions in the three regions is a characteristic and rather unusual feature of the present problem.

One rather surprising result, though, concerns the expression for the dimensionless shear stress at the wall, which on account of (2.6) (2.7) and (2.9) becomes

$$\tau_0 = \frac{1}{R}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{2\beta_1^{-\frac{1}{3}}}{9}(Rx)^{-\frac{5}{6}}\lambda^{\frac{1}{2}}(2+\lambda)^{\frac{2}{3}}(1-\lambda)^{\frac{1}{3}}. \tag{2.10}$$

This is seen to vanish both where $C \to \infty\,(\lambda \to 0)$† and when $C \to 0{\cdot}6192\,(\lambda \to 1)$, and in fact has a maximum at $\lambda = \frac{2}{3}$. Of course the vanishing of the shear stress as $C \to \infty$ is understandable since in that case the streamlines become vertical for $x > 0$. However, since, as in any conventional boundary layer, $(\partial u/\partial y)_{y=0}$ is $O(R^{\frac{1}{2}})$ when $C < 0{\cdot}6192$, the vanishing of (2.10) implies that the functional dependence of the wall shear stress on C must be very peculiar indeed when C lies near the critical value for blow off. This is confirmed in figure 4 where we have plotted the quantity $(Rx)^{\frac{5}{6}}\tau_0$ as a function of λ for $Rx = 10^6$. The curve for this figure was computed on the basis of (2.10) for $\lambda < 1$, while for

$$C < 0{\cdot}6192\,(\lambda > 1)$$

the expression
$$(Rx)^{\frac{5}{6}}\tau_0 = \tfrac{1}{4}(Rx)^{\frac{1}{3}}f''(0),$$

was used, where the quantity $f''(0)$ was obtained as a function of λ from Emmons & Leigh's (1954) numerical solution of the Blasius equation with the appropriate boundary conditions for blowing. The apparent singularity near $\lambda = 1$ should

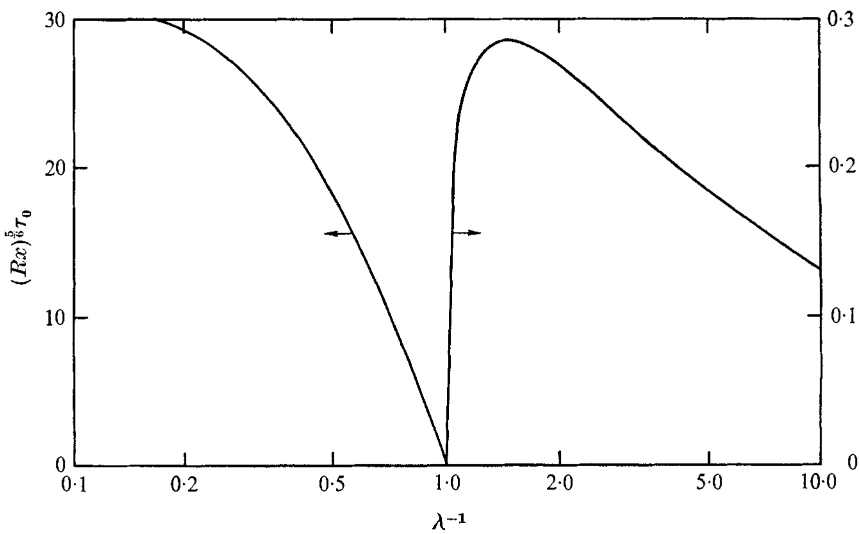

FIGURE 4. Wall shear stress as a function of $\lambda \equiv (\beta_1/C)^{\frac{2}{3}}$ for $Rx = 10^6$.

not be taken too seriously since, as was remarked earlier, the conventional boundary-layer result fails as $\lambda = 1$ from above while, as will be shown in the next section, the present solution remains valid when λ is close to 1 only if $(1-\lambda)^{\frac{1}{3}} \gg (Rx)^{-\frac{1}{6}}$. Nevertheless, the shape of the curve in figure 4 is surprising and therefore worth noting.

Returning now to our similarity solution, as given by (2.6), (2.7) and (2.9), we can easily show, by estimating the error in (2.5) arising from the absence of the viscous term, that it applies only as long as $(Rx)^{\frac{1}{6}} \gg 1$. This condition is of

† It is apparent from (2.1) that, as will be discussed somewhat more fully in the next section, the present analysis ceases to apply when C becomes of the same order of magnitude as $(Rx)^{\frac{1}{2}}$, i.e. we require that $\lambda(Rx)^{\frac{1}{3}} \gg 1$.

course rather restrictive and significantly limits the range of validity of our result. Hence, we proceed in the next section to construct an asymptotic expansion to the solution in terms of the small parameter $(Rx)^{-\frac{1}{6}}$.

3. Higher order analysis

As expected, the strong interdependence of the solutions in the three regions which is apparent in the asymptotic results of the previous section increases in complexity when the analysis is extended to higher order in Rx. Thus, as is generally the case with perturbation expansions of this type, a higher order solution in one region depends not only on the solutions of that same order in the other regions, but also, as will be described shortly, on the lower order terms. This interaction is further enhanced by the form of the expansion in each of the three regions in that, as the power of Rx is decreased by only $\frac{1}{6}$ at each successive step, a large number of terms are required in order to obtain even a moderate increase in the range of validity in terms of Rx of the asymptotic solution given earlier.

The fact that the expansion should proceed in this manner can be deduced from the observation that, as mentioned in § 2, the ratio of inertia to viscous forces in the blowing region is $O[(Rx)^{\frac{1}{6}}]$. This suggests that in order to achieve a proper balance of these forces in the higher order analysis the stream function in I should be expanded in powers of $(Rx)^{-\frac{1}{6}}$. Also, since the boundary conditions for the solution in each region are determined in part by matching the solutions in the overlapping domains between adjacent regions, the expansion of the stream function in II and III must also be in terms of $(Rx)^{-\frac{1}{6}}$.

Before proceeding with the higher order analysis, it is helpful to consider the equations describing the flow in regions II and III and the form of the corresponding solutions in more detail. Let us turn first to the boundary layer. The usual procedure here would be to employ a curvilinear co-ordinate system with one of the axes along the $\Psi = 0$ streamline, i.e. the streamline separating the injected fluid from that of the free stream. In the present problem, however, the use of such a curvilinear system would require the boundary-layer co-ordinates to be transformed into Cartesian co-ordinates prior to matching the solution in II with those in I and III, which are best handled within the Cartesian framework. This complication, although not severe, may be avoided by noting that since the thickness of the blowing region is only $O(R^{-\frac{1}{3}})$ the boundary-layer solution can be developed just as well using Cartesian co-ordinates. Hence we can express the leading term of the boundary-layer solution in the similar form:

$$\Psi_{\mathrm{II}}(x,\eta) = (x/R)^{\frac{1}{2}}f(\eta); \quad \eta \equiv (R/x)^{\frac{1}{2}}[y-\phi(x;R)], \tag{3.1}$$

where x, y are the Cartesian co-ordinates defined in figure 2 and $\phi(x; R)$ represents the position of the $\Psi = 0$ streamline which is given by (2.3) to a first approximation. Inserting the above expression for $\Psi_{\mathrm{II}}(x,\eta)$ into the x and y components of the Navier–Stokes equations leads to

$$2f'''(\eta)+f(\eta)f''(\eta) = 2x\,\partial P_{\mathrm{II}}/\partial x+O(\phi^2), \tag{3.2}$$

$$\partial P_{\mathrm{II}}/\partial \eta = -(x/R)^{\frac{1}{2}}\phi''(x;R)f'^2(\eta)+O(R^{-1}), \tag{3.3}$$

where $P_{\text{II}}(x,\eta)$ is the pressure in the boundary layer. Note that (3.3) is the same expression one would have obtained if boundary-layer co-ordinates had been used with one of the axes directed along the $\Psi = 0$ streamline, whose curvature is $\phi''(x;R)$. Evidently, since $\phi(x;R)$ is $O(R^{-\frac{1}{3}})$, the pressure drop across the boundary layer is $O(R^{-\frac{2}{3}})$. To first order then, $P_{\text{II}}(x,\eta) = 0$ and the problem reduces to that solved by Lock (1951).

In the free-stream region the first-order solution corresponds to flow over a thin body whose shape is given by $y = \phi(x;R)$. Putting $\Psi_{\text{III}}^{(1)} = R^{\frac{1}{3}}(\Psi_{\text{III}} - y)$, we have

$$\nabla^2\Psi_{\text{III}}^{(1)} = 0, \quad \text{with} \quad \Psi_{\text{III}}^{(1)}(x,0) = \begin{cases} 0 & (x < 0), \\ -\alpha_1 x^{\frac{2}{3}} & (x \geqslant 0), \end{cases}$$

$$\Psi_{\text{III}}^{(1)} \to 0 \quad \text{as} \quad y \to \infty, \quad x \to -\infty.$$

The solution of the above for the more general boundary condition along the positive x axis, $\Psi_{\text{III}}^{(1)}(x,0) = A_m x^m$, $-1 < m < 1$, which we include for future reference, is

$$\Psi_{\text{III}}^{(1)}(x,y) = A_m \frac{y}{\pi}\int_0^\infty \frac{z^m dz}{(x-z)^2+y^2},$$

or, in terms of the similarity variable $\xi = y/x$,

$$\Psi_{\text{III}}^{(1)}(x,\xi) = x^m t(\xi); \;\; t(\xi) = A_m \frac{\xi}{\pi}\int_0^\infty \frac{z^m dz}{(1-z)^2+\xi^2}. \tag{3.4}$$

Hence the stream function in III is given to a first approximation by

$$\Psi_{\text{III}}(x,\xi) = x\xi + (x^{\frac{2}{3}}/R^{\frac{1}{3}})\, t(\xi), \tag{3.5}$$

where $t(\xi)$ is obtained from (3.4) with $m = \frac{2}{3}$ and $A_m = -\alpha_1$. The complete solution to the first-order problem is then represented by (3.5), (3.1) and (2.6).

Higher order solutions can now be generated by formally expanding the stream function in each of the three regions in powers of $(Rx)^{-\frac{1}{3}}$. Not too surprisingly perhaps, in view of the requirement that the solution remain independent of the choice of length scale l even in the higher order analysis, each term of the series will retain the self-similar form of the corresponding first-order solution. Of course, by analogy with the classical case of flow past a semi-infinite flat plate without blowing, we would expect that eventually terms involving logarithms and eigensolutions would appear in the expansions, thereby complicating the analysis. As we shall presently demonstrate, however, these additional terms do not begin to appear until the fourth-order solution is considered. Consequently, the appropriate expansions are

$$\begin{aligned}\Psi_{\text{I}}(x,\zeta) = -2C(x/R)^{\frac{1}{2}}\{&g_1(\zeta) + (Rx)^{-\frac{1}{6}}g_2(\zeta) + (Rx)^{-\frac{1}{3}}g_3(\zeta) \\ &+ (Rx)^{-\frac{1}{2}}[g_4(\zeta)\ln(Rx) + g_5(\zeta)] + \ldots\},\end{aligned} \tag{3.6}$$

$$\begin{aligned}\Psi_{\text{II}}(x,\eta) = (x/R)^{\frac{1}{2}}\{&f_1(\eta) + (Rx)^{-\frac{1}{6}}f_2(\eta) + (Rx)^{-\frac{1}{3}}f_3(\eta) \\ &+ (Rx)^{-\frac{1}{2}}[f_4(\eta)\ln(Rx) + f_5(\eta)] + \ldots\},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\Psi_{\text{III}}(x,\xi) = x\xi + (x^{\frac{2}{3}}/R^{\frac{1}{3}})\{&t_1(\xi) + (Rx)^{-\frac{1}{6}}t_2(\xi) + (Rx)^{-\frac{1}{3}}t_3(\xi) \\ &+ (Rx)^{-\frac{1}{2}}[t_4(\xi)\ln(Rx) + t_5(\xi)] + \ldots\},\end{aligned} \tag{3.8}$$

in which $f_1(\eta)$, $g_1(\zeta)$, and $t_1(\xi)$ now represent the first-order solutions. Also, because these higher order terms will produce additional corrections to the position of the $\Psi = 0$ streamline, the function $\phi(x;R)$, which appears in the expression for η, must similarly be expanded, i.e.

$$\phi(x;R) = (x^{\frac{2}{3}}/R^{\frac{1}{3}})\{\alpha_1 + \alpha_2 (Rx)^{-\frac{1}{6}} + \alpha_3 (Rx)^{-\frac{1}{3}} + (Rx)^{-\frac{1}{2}}[\alpha_4 \ln(Rx) + \alpha_5] + \ldots\}. \quad (3.9)$$

It will be possible to uniquely determine the first four terms in (3.6)–(3.9) through straightforward application of standard expansion and matching techniques. However, $f_4(\eta)$, the fourth term in (3.7), plays the role of an eigen-solution to the appropriate boundary-layer equation for $f_5(\eta)$ hence the latter will contain an unknown constant. In addition, since this indeterminacy will be transferred to the expressions for $g_5(\zeta)$, $t_5(\xi)$ and α_5 through the matching of (3.7) to (3.6) and (3.8), the solutions to (3.6)–(3.9) cannot be obtained uniquely beyond four terms without considering the details of the motion near the leading edge. Consequently our analysis will be terminated at this point.

The form of (3.6)–(3.9) allows us next to determine the pressure in I and II before actually generating the higher order equations for the stream functions by expanding the pressure in each region in terms of $(Rx)^{-\frac{1}{6}}$ and then matching the corresponding series in the areas of overlap. We begin in the blowing region where the pressure through fourth order is a function of Rx only since the pressure drop across I is $O(R^{-1})$. Hence,

$$P_{\mathrm{I}}(x,\zeta) = p_1 (Rx)^{-\frac{1}{3}} + p_2 (Rx)^{-\frac{1}{2}} + p_3 (Rx)^{-\frac{2}{3}} + p_4 (Rx)^{-\frac{5}{6}} \ln(Rx) + O[(Rx)^{-\frac{5}{6}}], \quad (3.10)$$

where the p_n's, $n = 1-4$, are constants to be determined. Furthermore, because the pressure drop is $O[(Rx)^{-\frac{5}{6}}]$ across the boundary layer, the first four terms in the expansion for $P_{\mathrm{II}}(x,\eta)$ can be immediately matched to (3.10), yielding

$$P_{\mathrm{II}}(x,\eta) = p_1 (Rx)^{-\frac{1}{3}} + p_2 (Rx)^{-\frac{1}{2}} + p_3 (Rx)^{-\frac{2}{3}} + p_4 (Rx)^{-\frac{5}{6}} \ln(Rx) + O[(Rx)^{-\frac{5}{6}}]. \quad (3.11)$$

In region III, Bernoulli's equation applies, thus

$$P_{\mathrm{III}}(x,\xi) = \frac{1}{2} - \frac{1}{2}\left\{\left[\frac{\partial \Psi_{\mathrm{III}}}{\partial x}\right]^2 + \frac{1}{x^2}\left[\frac{\partial \Psi_{\mathrm{III}}}{\partial \xi}\right]^2\right\}.$$

For matching purposes, we require the form of the above as $\xi \to 0$. This can be obtained readily using (3.8) and (3.9), and replacing ξ by its equivalent

$$(x^{-1}\phi + (Rx)^{-\frac{1}{2}}\eta),$$

so that

$$P_{\mathrm{III}}(x,\xi) \xrightarrow[\xi \to 0]{} -(Rx)^{-\frac{1}{3}} t_1'(0) - (Rx)^{-\frac{2}{3}}[t_3'(0) + \alpha_1 t_1''(0) + \tfrac{2}{9} t_1^2(0) + \tfrac{1}{2} t_1'^2(0)]$$
$$- (Rx)^{-\frac{5}{6}} \ln(Rx)\, t_4'(0) + O[(Rx)^{-\frac{5}{6}}], \quad (3.12)$$

where terms involving $t_2'(0)$ have been omitted since $t_2'(0)$ represents the correction to the pressure along the surface of a parabola which is known to be zero. In addition, it can easily be shown from (3.4) that

$$\left.\begin{aligned} t_n(0) &= A_m, \\ t_n'(0) &= -mA_m \cot m\pi \quad (-1 < m < 1), \\ t_n''(0) &= m(1-m)A_m, \end{aligned}\right\} \quad (3.13)$$

which yields, for the first-order solution,

$$t_1'(0) = -2\alpha_1/3^{\frac{4}{3}}, \quad t_1''(0) = -\tfrac{2}{9}\alpha_1.$$

Finally, by matching (3.10) and (3.12) to the form of (3.11) as $\eta \to \pm\infty$, we arrive at

$$\left.\begin{aligned} p_1 &= 2\alpha_1/3^{\frac{4}{3}}, & p_2 &= 0, \\ p_3 &= -t_3'(0)-(2/27)\alpha_1^2, & p_4 &= -t_4'(0), \end{aligned}\right\} \tag{3.14}$$

where the unknown terms in p_3 and p_4 will be determined from the higher order solutions in the free stream.

Having resolved the form of the pressure expansions in each region, we now proceed to construct the equations for the higher order similarity functions. Beginning with the boundary layer we obtain the higher order equations by substituting (3.7) and (3.11) into the x component of the Navier–Stokes equations and equating terms with like powers of Rx. Thus,

$$\left.\begin{aligned} 6f_2''' + 3f_1f_2'' + f_1'f_2' + 2f_1''f_2 &= 0, \\ 6f_3''' + 3f_1f_3'' + 2f_1'f_3' + f_1''f_3 &= -2p_1 - f_2'^2 - 2f_2f_2'', \end{aligned}\right\} \tag{3.15}$$

and

$$6f_4''' + 3f_1f_4'' + 3f_1'f_4' = 0, \tag{3.16}$$

$$6f_5''' + 3f_1f_5'' + 3f_1'f_5' = -2f_2f_3'' - 3f_2'f_3' - f_2''f_3 + 6[f_1'f_4' - f_1''f_4], \tag{3.17}$$

where $f_4(\eta)$ is an eigensolution of (3.17). Since η is by definition equal to zero along the $\Psi = 0$ streamline

$$f_2(0) = f_3(0) = f_4(0) = f_5(0) = 0. \tag{3.18}$$

The two remaining boundary conditions for each of the above equations will be provided by the requirement that $f_n'(\eta)$ must match, as $\eta \to \pm\infty$, with the corresponding solutions in I and III respectively.

In considering the solution to (3.16) with homogeneous boundary conditions it is clear that the eigenfunction $f_4(\eta)$ is simply

$$f_4(\eta) = K[f_1'(\eta)/f_1'(0) - 1], \tag{3.19}$$

in which K is a constant. This eigensolution must be included as a logarithmic term in the expansion (3.7) since, in its absence, it would not be possible to obtain a solution to (3.17) satisfying the required boundary conditions. Thus the eigenfunction $f_4(\eta)$ plays an important role in the fourth-order boundary-layer solution in that, by appearing in the non-homogeneous part of (3.17), it allows $f_5(\eta)$ to assume the proper form as $\eta \to -\infty$. To determine K, we integrate (3.17) once to obtain

$$f_5'' + \tfrac{1}{2}f_1f_5' = (K/f_1'(0))f_1' - \tfrac{1}{3}f_2f_3' - \tfrac{1}{6}f_2'f_3 + \text{constant}, \tag{3.20}$$

from which K can be found, given the asymptotic forms of the f_n's as $\eta \to \pm\infty$. Of course the appearance of $f_4(\eta)$ in the analysis precludes the uniqueness of $f_5(\eta)$ since $f_4(\eta)$ multiplied by any constant is also a solution to the homogeneous part of (3.17).

Turning next to the blowing region, we insert (3.6) and (3.10) into the x

momentum equation to obtain the following expressions for the similarity functions $g_n(\zeta)$:

$$\left.\begin{aligned} &3g_1g_2''+3g_1'g_2'+2g_1''g_2 = (3/C)g_1''',\\ &3g_1g_3''+4g_1'g_3'+g_1''g_3 = -(p_3/C^2)+(3/C)g_2'''-2g_2'^2-2g_2g_2'',\\ &3g_1g_4''+5g_1'g_4' = -\tfrac{5}{4}p_4/C^2, \end{aligned}\right\} \tag{3.21}$$

with
$$g_n(0) = g_n'(0) = 0, \quad (n = 2, 3, 4). \tag{3.22}$$

We will not be concerned here with the equation for $g_5(\zeta)$ since its solution contains an unknown constant arising through the matching of (3.6) to (3.7).

In view of the fact that the equations (3.21) are second order, the boundary conditions (3.22) are sufficient to insure a unique solution without reference to the adjacent boundary-layer region. For this reason matching of the stream function expansions for I and II in their domain of overlap will give rise to additional conditions from which the α_n's will be determined. However, since, as will be seen in (3.30), each p_n is a function of the corresponding α_n which prior to this matching is unknown, the solutions to the equations in (3.21) will be expressed in the form

$$\left.\begin{aligned} g_2(\zeta) &= g_2(\zeta),\\ g_3(\zeta) &= \alpha_3 g_{31}(\zeta)+g_{32}(\zeta),\\ g_4(\zeta) &= \alpha_4 g_{41}(\zeta), \end{aligned}\right\} \tag{3.23}$$

where g_{n1} and g_{n2} are independent of the corresponding α_n.

Finally, in the free-stream region it is apparent that since each higher order term satisfies Laplace's equation with the appropriate boundary conditions along the x axis, the nth-order problem for $n = 2$ and 3 becomes

$$\nabla^2\Psi_{\mathrm{III}}^{(n)} = 0; \quad \Psi_{\mathrm{III}}^{(n)}(x,0) = \begin{cases} 0 & (x<0)\\ A_n x^{(5-n)/6} & (x \geqslant 0)\end{cases},$$
$$\Psi_{\mathrm{III}}^{(n)} \to 0 \quad \text{as} \quad x \to -\infty, \quad \xi \to \infty,$$

where $\Psi_{\mathrm{III}}^{(n)}(x,\xi)$ refers to the term in (3.8) containing $t_n(\xi)$, and A_n is a constant to be determined through matching with the boundary-layer solution. For $n = 4$, the problem is the same except that $\ln(Rx)$ must be included in the expression for $\Psi_{\mathrm{III}}^{(n)}(x,0)$. In view of (3.4),

$$t_n(\xi) = A_n\frac{\xi}{\pi}\int_0^\infty \frac{z^{(5-n)/6}\,dz}{(1-z)^2+\xi^2}, \quad (n = 2, 3, 4). \tag{3.24}$$

Again, as was done previously with the solution in region I, we shall terminate the analysis in III after obtaining four terms since the higher order terms cannot be uniquely determined.

To complete the solution we will now determine the remaining unknown boundary conditions and constants by matching (3.8), as $\xi \to 0$, and (3.6), as $\zeta \to \alpha_1$, to the asymptotic forms of (3.7) as $\eta \to \pm\infty$, respectively. Expanding first the stream function in the blowing region as $\zeta \to \alpha_1$, we have that

$$\left.\begin{aligned} \Psi_{\mathrm{I}}(x,\zeta) \xrightarrow[\zeta\to\alpha_1]{} &-2C(x/R)^{\frac{1}{2}}\{g_1(\alpha_1)+(\zeta-\alpha_1)g_1'(\alpha_1)+\tfrac{1}{2}(\zeta-\alpha_1)^2 g_1''(\alpha_1)\\ &+\tfrac{1}{6}(\zeta-\alpha_1)^3 g_1'''(\alpha_1)+O[(\zeta-\alpha_1)^4]+(Rx)^{-\frac{1}{6}}[g_2(\alpha_1)+(\zeta-\alpha_1)g_2'(\alpha_1)\\ &+\tfrac{1}{2}(\zeta-\alpha_1)^2 g_2''(\alpha_1)+O[(\zeta-\alpha_1)^3]]+(Rx)^{-\frac{1}{3}}[g_3(\alpha_1)+(\zeta-\alpha_1)g_3'(\alpha_1)\\ &+O[(\zeta-\alpha_1)^2]]+(Rx)^{-\frac{1}{2}}\ln(Rx)\,[g_4(\alpha_1)+O(\zeta-\alpha_1)]+O[(Rx)^{-\frac{1}{2}}]\}. \end{aligned}\right\} \tag{3.25}$$

Similarly, for the free-stream region

$$\Psi_{\mathrm{III}}(x,\xi) \xrightarrow[\xi\to 0]{} (x^{\frac{2}{3}}/R^{\frac{1}{3}})\{-\alpha_1-(2\alpha_1/3^{\frac{3}{2}})\xi+A_2(Rx)^{-\frac{1}{6}}+A_3(Rx)^{-\frac{1}{3}} + A_4(Rx)^{-\frac{1}{2}}\ln(Rx)+O[(Rx)^{-\frac{1}{2}}]\}. \quad (3.26)$$

For the solution in the boundary layer, as $\eta\to\infty$,

$$f_n(\eta) \xrightarrow[\eta\to\infty]{} \Gamma_n\eta-\gamma_n = \Gamma_n(Rx)^{\frac{1}{2}}(\xi-x^{-1}\phi)-\gamma_n, \quad (3.27)$$

where Γ_n and γ_n are constants. Therefore by substituting (3.27) into (3.7) and matching term by term to (3.26) we obtain

$$\Gamma_2=\Gamma_4=0, \quad \Gamma_3=-p_1, \quad (3.28)$$

and
$$A_2=-(\alpha_2+\gamma_1), \quad A_3=-(\alpha_3+\gamma_2-p_1\alpha_1), \quad A_4=-\alpha_4. \quad (3.29)$$

The expression (3.8), with (3.24) and (3.29), now provides the solution for $\Psi_{\mathrm{III}}(x,\xi)$ to $O[(Rx)^{-\frac{5}{6}}\ln(Rx)]$. In addition, using (3.29) in (3.13) with

$$A_m=A_n, \quad m=\tfrac{1}{6}(5-n),$$

and substituting the resulting expressions into (3.14) yields the following relations for the pressure terms:

$$\left.\begin{aligned} p_1 &= 2\alpha_1/3^{\frac{3}{2}}, & p_2 &= 0,\\ p_3 &= -(1/3^{\frac{3}{2}})(\alpha_3+\gamma_2), & p_4 &= -\tfrac{1}{6}3^{\frac{1}{2}}\alpha_4. \end{aligned}\right\} \quad (3.30)$$

Let us now consider the matching of the stream functions in I and II in their region of overlap. Since η can be expressed in terms of $(\zeta-\alpha_1)$ as

$$\eta=(Rx)^{\frac{1}{6}}\{(\zeta-\alpha_1)-\alpha_2(Rx)^{-\frac{1}{6}}-\alpha_3(Rx)^{-\frac{1}{3}}-\alpha_4(Rx)^{-\frac{1}{2}}\ln(Rx)-\ldots\}, \quad (3.31)$$

it is apparent that in order to achieve a proper matching of the terms in (3.25) with the corresponding terms in (3.7) as $\eta\to-\infty$ we must have that

$$f_n(\eta) \xrightarrow[\eta\to-\infty]{} B_3^{(n)}\eta^3+B_3^{(n)}\eta^2+B_1^{(n)}\eta-2\beta_n \quad (n=2,3,4,5), \quad (3.32)$$

where $B_1^{(n)}$, $B_2^{(n)}$, $B_3^{(n)}$ and β_n are constants and

$$B_2^{(2)}=B_3^{(2)}=B_3^{(3)}=B_1^{(4)}=B_2^{(4)}=B_3^{(4)}=0.$$

Also, by direct substitution of (3.32) into (3.15) and (3.16) it can be easily verified that (3.32) correctly represents the form of $f_n(\eta)$ as $\eta\to-\infty$. Thus, inserting (3.32) and (3.31) into (3.7) and matching this expansion to (3.25) determines the constants in (3.32),

$$\left.\begin{aligned} &B_1^{(2)}=-2Cg_1'(\alpha_1), \quad B_1^{(3)}=-2C[\alpha_2 g_1''(\alpha_1)+g_2'(\alpha_1)], \quad B_2^{(3)}=-Cg_1''(\alpha_1),\\ &B_1^{(5)}=2\alpha_2B_2^{(5)}-3\alpha_2^2B_3^{(5)}+2\alpha_3B_2^{(3)}-2Cg_3'(\alpha_1),\\ &B_2^{(5)}=-C[g_2''(\alpha_1)+\alpha_2g_1'''(\alpha_1)], \quad B_3^{(5)}=-\tfrac{1}{3}Cg_1'''(\alpha_1), \end{aligned}\right\} \quad (3.33)$$

in addition to providing the following expressions for the α_n's:

$$\left.\begin{aligned} \alpha_2 &= g_1'^{-1}(\alpha_1)\{\beta_2/C-g_2(\alpha_1)\},\\ \alpha_3 &= [g_1'(\alpha_1)+g_{31}(\alpha_1)]^{-1}\{\beta_3/C-\tfrac{1}{2}\alpha_2^2g_1''(\alpha_1)-\alpha_2g_2'(\alpha_1)-g_{32}(\alpha_1)\},\\ \alpha_4 &= -(\beta_4/C)[g_1'(\alpha_1)+g_{41}(\alpha_1)]^{-1}. \end{aligned}\right\} \quad (3.34)$$

Finally, on account of (3.27) and (3.32), the boundary conditions for the functions $f_n'(\eta)$ as $\eta \to \pm\infty$, which are required for the integration of (3.15), are

$$\left.\begin{aligned} &f_2'(\infty) = 0, \quad f_3'(\infty) = \Gamma_3, \\ &f_2'(-\infty) = B_1^{(2)}, \quad f_3'(\eta) \xrightarrow[\eta\to-\infty]{} 2B_2^{(3)}\eta + B_1^{(3)}, \end{aligned}\right\} \tag{3.35}$$

with the constants given by (3.28) and (3.33).

Owing to the strong interdependence of the terms in (3.6), (3.7) and (3.8) evaluation of the similar functions and the related constants must proceed in a prescribed order. Thus, if we consider the nth-order analysis the calculations will begin in the boundary layer where, for $n = 2, 3$, the boundary conditions (3.19) and (3.35) and the pressure term (3.30) for the appropriate equation in (3.15) are determined entirely from the lower order solutions in I and III. Integration of this pair of linear equations provides then values for the constants β_n and γ_n. For $n = 4$, the eigenfunction $f_4(\eta)$ is given by (3.19) and thus $\beta_4 = \frac{1}{2}K$. In order to calculate K, we must then evaluate (3.20) in conjunction with (3.27) and (3.32), which yields

$$\beta_4 = \tfrac{1}{2}K = \tfrac{1}{6}[3\beta_1 B_1^{(5)} + 2\beta_2 B_1^{(3)} + \beta_3 B_1^{(2)} + \gamma_2 p_1 - 6B_1^{(5)}]. \tag{3.36}$$

Continuing the nth-order analysis we move next to the blowing region where the appropriate equation for $g_n(\zeta)$ in (3.21) is integrated, subject to (3.22) and (3.30), to produce the nth-order term in (3.23), which in turn is used in (3.34) to determine α_n. With α_n known (3.24) and (3.29) then represent the solution for $t_n(\xi)$, the nth-order term for the expansion in the free-stream region. This completes the nth-order calculations and provides the information necessary to begin the next order analysis which will proceed in the same manner.

λ	α_1	α_2	α_3	α_4	p_1	p_2	p_3	p_4
0·9	1·320	2·151	−7·997	$-7{\cdot}2\times10^{-4}$	0·5080	0	1·584	$2{\cdot}1\times10^{-4}$
0·8	1·827	2·039	−9·929	$-1{\cdot}26\times10^{-2}$	0·7033	0	1·990	$3{\cdot}65\times10^{-3}$
0·7	2·333	1·943	−12·45	$3{\cdot}11\times10^{-2}$	0·8981	0	2·513	$-8{\cdot}98\times10^{-3}$
0·6	2·922	1·864	−15·70	$4{\cdot}38\times10^{-2}$	1·125	0	3·185	$-1{\cdot}27\times10^{-2}$
0·5	3·679	1·803	−20·15	0·1053	1·416	0	4·104	$-3{\cdot}04\times10^{-2}$
0·4	4·756	1·761	−26·77	0·3418	1·831	0	5·466	$-9{\cdot}87\times10^{-2}$
0·3	6·489	1·739	−37·8	1·58	2·498	0	7·73	−0·455
0·2	9·880	1·739	−60·0	12·4	3·803	0	12·2	−3·60
0·1	19·92	1·74	−125	110	7·668	0	25·5	−32·5

TABLE 1. Coefficients in the expansions for the pressure and the position of the $\Psi = 0$ streamline as a function of λ

In order to evaluate the additional information gained from the higher order analysis, the calculations, as outlined above, were performed numerically to determine the coefficients appearing in the expressions for the pressure and the position of the streamline separating the injected fluid from the free stream. The results of these computations are tabulated as a function of λ in table 1 for $n = 1-4$. Clearly, whereas the second-order coefficients are of smaller magnitude than the corresponding first-order ones (except for λ very close to unity), the third-order coefficients are relatively large. Thus, the ratio of α_3 to α_1 is about

$-5\cdot 5$ while p_3 is about three times larger than p_1. However, in turning to the fourth-order coefficients we see that this trend is not continued, for, although α_4 and p_4 are large near $\lambda = 0$ they both decrease rapidly in magnitude as λ is increased and become very small when $\lambda \to 1$. It is also interesting to note that at $\lambda = 0\cdot 7613$, corresponding to $C = \frac{3}{2}\beta_1$, the denominator in (3.34), the expression for α_4, vanishes, implying that the expansions as written become singular. However, since in (3.34) β_4 is only $O(10^{-3})$ at this value of λ, the effect of this singularity becomes significant only when λ is very close to $0\cdot 7613$. Nevertheless, to complete the analysis it is necessary to remove this singularity by slightly altering the form of the expansions at this particular point. This can be accomplished through the addition of a $\ln^2(Rx)$ term to the fourth-order parts of (3.6), (3.9), and (3.10) which now become respectively

$$-(2C/R)\,[g_{4e}(\zeta)\ln^2(Rx)+g_4(\zeta)\ln(Rx)+g_5(\zeta)],$$

$$(x^{\frac{1}{6}}/R^{\frac{5}{6}})\,[\alpha_{4e}\ln^2(Rx)+\alpha_4\ln(Rx)+\alpha_5],$$

and
$$(Rx)^{-\frac{5}{6}}[p_{4e}\ln^2(Rx)+p_4\ln(Rx)+p_5].$$

By reworking the fourth-order analysis it can be shown now that, indeed for $\lambda \neq 0\cdot 7613$, $g_{4e}(\zeta) \equiv \alpha_{4e} \equiv p_{4e} \equiv 0$. On the other hand at $\lambda = 0\cdot 7613$ an additional term involving α_{4e} appears in the numerator of (3.34). Therefore, by choosing α_{4e} in such a way that this numerator also vanishes, thereby removing the singularity in (3.34), we obtain

$$\alpha_{4e} = 5\cdot 83\times 10^{-5}, \quad p_{4e} = -1\cdot 68\times 10^{-5}.$$

In principle, α_4 could be determined in a similar fashion by removing the singularity in the expression for α_5 analogous to (3.34). However, since, as mentioned following (3.22), the fifth-order solutions contain an unknown constant such an additional calculation would be rather pointless at this stage.

It has already been mentioned in § 2 that when λ is near unity the solution is valid only for $(Rx)^{\frac{1}{6}} \gg (1-\lambda)^{-\frac{1}{3}}$. This restriction becomes apparent if we consider the relative magnitudes of α_1 and α_2. Clearly, from (2.9), $\alpha_1 \to O[(1-\lambda)^{\frac{1}{3}}]$ while, as can easily be proved analytically or seen in table 1, α_2 remains $O(1)$. Thus, unless $(Rx)^{\frac{1}{6}} \gg (1-\lambda)^{-\frac{1}{3}}$, the leading term in the expansions will not dominate the series and the assumptions involved in the first-order analysis will no longer be valid. It is also evident from table 1 that the solution breaks down at the other extreme when the rate of fluid injection becomes very large, i.e. as $\lambda \to 0$. This is to be expected, however, since, if λ becomes of the same numerical order of magnitude as $(Rx)^{-\frac{1}{3}}$, C will become numerically $O[(Rx)^{\frac{1}{2}}]$ with the result that v_0 and ϕ will become $O(1)$ in magnitude. This last condition corresponds to the case of massive blowing, treated by Cole & Aroesti (1968), which is outside the scope of the present work.

4. Structure of the flow for more general blowing velocity profiles

Up to this point our analysis has been mainly restricted to rates of fluid injection proportional to $(Rx)^{-\frac{1}{2}}$. We now examine other blowing velocity profiles which also lead to blow off of the boundary layer. A well-known example in this category is the case of uniform injection along the surface of the plate, which causes the boundary layer to become detached at a finite distance downstream from the leading edge. In considering this uniform blowing profile, Lew & Fanucci (1955) numerically treated the boundary layer upstream of separation and found that the skin friction decreased rapidly with increasing distance from the leading edge. Catherall *et al.* (1965) extended these computations to the point of separation and then analytically examined the form of the boundary-layer solution in this vicinity. Downstream of separation, however, the structure of the flow does not appear to have been established.

In this section we consider the position of the $\Psi = 0$ streamline downstream of separation, first for general blowing profiles and then for profiles of the form $v_0 \sim R^{-\frac{1}{2}}x^{m-1}$, $0 < m \leqslant 1$, placing special emphasis on the case of uniform injection. To begin with, since we shall retain the requirement that $v_0(x)$ be $O(R^{-\frac{1}{2}})$, it is easy to see from the arguments presented in § 2 that the region containing the injected fluid will become $O(R^{-\frac{1}{2}})$ in thickness downstream of the point of detachment. Thus, to a first approximation, this blowing region will again be inviscid with the pressure a function of x only. With this in mind, an expression for $\phi(x;R)$ can be derived following the procedure of Cole & Aroesti (1968) mentioned previously, except that the stream function at the upper edge of the blowing region must now be matched to the adjacent boundary-layer solution, thereby yielding

$$\phi(x;R) = 2^{-\frac{1}{2}} \int_{\Psi_0(x)}^{-2\beta_1(x/R)^{\frac{1}{2}}} \frac{d\Psi}{[P(x^*) - P(x)]^{\frac{1}{2}}}. \tag{4.1}$$

Here, $P(x)$ is given in terms of $\phi'(x;R)$ from thin airfoil theory and $x^*(\Psi)$ is the inverse function of $\Psi_0(x)$, the stream function at $y = 0$. Note that in deriving the upper limit for the integral in (4.1) Lock's (1951) solution has been used to describe the boundary layer between the inviscid blowing region and the free stream since, to a first approximation, the velocity in III remains unity while that in I may be neglected. However, this can be justified over the entire region of separated flow (except in the very close proximity of the separation point) only when detachment occurs at the leading edge; otherwise, if the point of breakaway lies at some finite distance x_s from the leading edge the similar form of Lock's solution will become valid only asymptotically for $x \gg x_s$, owing to the influence of the non-uniform 'initial' velocity profile at $x = x_s$. Apart from this restriction (4.1) is completely general and thus $\phi(x;R)$ can be determined in principle from (4.1) through an iterative process for arbitrary $\Psi_0(x)$.

In order to examine the structure of the flow downstream from detachment we shall focus on the class of blowing velocity profiles of the form

$$\Psi_0(x) = -R^{-\frac{1}{2}}x^m, \quad 0 < m \leqslant 1.$$

Here, for $m \neq \frac{1}{2}$, it is unnecessary to include a constant multiplying x^m since it can be eliminated by a suitable choice of the characteristic length. Of course for $m = \frac{1}{2}$ the results of the previous sections apply. Although our primary interest lies in the case of uniform injection, we shall presently show that if m is set equal to 1 the integral in (4.1) becomes undefined as $x \to \infty$. Thus, we begin by considering blowing profiles corresponding to $\frac{1}{2} < m < 1$ with a view to eventually letting $m \to 1$.

We first note that for blowing velocity profiles belonging to the above category the boundary layer will not separate from the surface of the plate until some distance downstream of the leading edge since it is capable of entraining fluid at a rate proportional to $x^{-\frac{1}{3}}$. As a result (4.1) represents the position of the $\Psi = 0$ streamline only for $x \geqslant x_s$. Furthermore, for $x \gg x_s$ the flow structure within the blowing region should be self-similar to a first approximation since with increasing distance downstream the flow should depend less and less on events occurring near the leading edge. Thus, to construct this similarity solution valid far downstream of the point of detachment, we assume that $\phi(x, R)$ is the form

$$\phi(x,R) = R^{-\frac{1}{3}}\{\alpha_{1m}x^{n_1} + \alpha_{2m}x^{n_2} + \ldots\} + \ldots \quad \text{for} \quad x \gg x_s, \tag{4.2}$$

where $n_1 > n_2 > \ldots$. Then, from thin airfoil theory,

$$P(x) = -R^{-\frac{1}{3}}\{\alpha_{1m}n_1 \cot(n_1\pi)\, x^{n_1-1} + \alpha_{2m}n_2 \cot(n_2\pi)\, x^{n_2-1} + \ldots\}. \tag{4.3}$$

Using the first term in (4.3) it is now possible to determine α_{1m} through a straightforward integration of (4.1). Before accomplishing this, however, we shall first develop the similarity solution in more detail in order to demonstrate that the similarity hypothesis does, in fact, yield a self-consistent solution for $x \gg x_s$. To begin with, we observe that since $\Psi_{\mathrm{I}} \sim x^m$ at $y = 0$, and $\Psi_{\mathrm{I}} \sim x^{\frac{1}{2}}$ at $y = \phi$, the expression for Ψ_{I} must be given by

$$\Psi_{\mathrm{I}}(x,\zeta) = -R^{-\frac{1}{2}}\{x^m g_{11}(\zeta) + x^{\frac{1}{2}} g_{12}(\zeta) + \ldots\} + \ldots, \tag{4.4}$$

where $\zeta = R^{\frac{1}{3}}x^{-n_1}y$. Physically, this expression reflects the fact that, for $x \gg x_s$, the rate of injection at $y = 0$ far exceeds the entrainment capabilities of the free shear layer located at $y = \phi$. Thus, to a first approximation, all the injected fluid is swept downstream. The second term in (4.4) then serves to satisfy the entrainment requirements of the shear layer.

By substituting (4.3) and (4.4) into the x momentum equation and balancing the inertia and pressure terms, we next obtain

$$n_1 = \tfrac{1}{3}(2m+1), \quad n_2 = \tfrac{1}{3}(\tfrac{5}{2} - m) \tag{4.5}$$

plus the following differential equations for $g_{11}(\zeta)$ and $g_{12}(\zeta)$:

$$mg_{11}g_{11}'' - (m-n_1)\, g_{11}'^2 + (n_1 - 1)\, n_1\alpha_{1m} \cot(n_1\pi) = 0$$

and

$$mg_{11}g_{12}'' - (m - 2n_1 + \tfrac{1}{2})\, g_{11}'g_{12}' + \tfrac{1}{2}g_{11}''g_{12} + (n_2 - 1)\, n_2\alpha_{2m}\cot(n_2\pi) = 0, \tag{4.6}$$

with boundary conditions

$$\left.\begin{aligned} g_{11}(0) &= 1, \quad g_{11}'(0) = g_{11}(\alpha_{1m}) = 0 \\ g_{12}(0) &= g_{12}'(0) = 0, \quad g_{12}(\alpha_{1m}) = 2\beta_1. \end{aligned}\right\} \tag{4.7}$$

Since only two boundary conditions are required for the integration of each equation in (4.6), the third condition for g_{11} and g_{12} in (4.7) allows one to evaluate the unknown constants α_{1m} and α_{2m}.

Having thus outlined the nature of the similarity solution, we now proceed with the calculation of α_{1m}. By integrating the first equation in (4.5) subject to (4.6) we obtain

$$\alpha_{1m}^{\frac{3}{2}} = \frac{m}{1-n_1}[-2n_1 \cot(n_1\pi)]^{-\frac{1}{2}} \int_1^\infty \frac{z^{(2+m)/(2m-2)}\,dz}{(z-1)^{\frac{1}{2}}}, \tag{4.8}$$

an expression which could also have been derived directly from (4.1) (note that, as mentioned earlier, this integral is undefined for $m = 1$). Therefore, upon evaluating the above integral and making use of (4.5), we arrive at

$$\alpha_m^{\frac{3}{2}} = \{-(m+\tfrac{1}{2})\cot[\tfrac{2}{3}(m+\tfrac{1}{2})\pi]\}^{-\frac{1}{2}} \frac{3^{\frac{3}{2}}\pi^{\frac{1}{2}}m}{4(1-m)} \frac{\Gamma[(m+\frac{1}{2})/(1-m)]}{\Gamma[(1+\frac{1}{2}m)/(1-m)]}. \tag{4.9}$$

For the case of constant blowing the above reduces to

$$\alpha_{m\to 1} = (\pi^{\frac{2}{3}}/2^{\frac{1}{3}}) + O(1-m) \tag{4.10}$$

while, from (4.5), $n = 1$. Since the latter indicates that the inviscid blowing region takes on the shape of a wedge, no solution can exist for the corresponding external flow if the plate is truly semi-infinite. For this reason, the problem of uniform blowing has meaning only for flat plates of finite length. Nevertheless, if the plate length is large compared to x_s, (4.10) will remain valid over the major part of the downstream portion of the plate and the position of the $\Psi = 0$ streamline will be given there by

$$\phi(x;R) = (\pi^2/2R)^{\frac{1}{3}}x. \tag{4.11}$$

For blowing velocity profiles corresponding to $0 < m < \frac{1}{2}$ the structure of the separated region differs significantly from that described above for $\frac{1}{2} < m \leqslant 1$. Near the leading edge fluid is injected into the flow with a velocity greater than $\beta_1(Rx)^{-\frac{1}{2}}$; consequently, the boundary layer is immediately 'blown away' from the plate. However, since entrainment into the boundary layer from below can exceed the blowing rate downstream, the boundary layer must eventually re-attach to the surface of the plate. Furthermore, from simple entrainment considerations it is evident that this re-attachment must occur at that value of x for which

$$\Psi_0(x) = -2\beta_1(x/R)^{\frac{1}{2}}. \tag{4.12}$$

Although in principle it should be possible to compute the shape of the separated flow region from (4.1) this will not be attempted here.

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REFERENCES

ACRIVOS, A. 1962 The asymptotic form of the laminar boundary layer mass-transfer rate for large interfacial velocities. *J. Fluid Mech.* **12**, 337.

CATHERALL, D., STEWARTSON, K. & WILLIAMS, P. G. 1965 Viscous flow past a flat plate with uniform injection. *Proc. Roy. Soc.* A **284**, 370.

COLE, J. D. & AROESTI, J. 1968 The blowhard problem—inviscid flows with surface injection. *Int. J. Heat Mass Transfer*, **11**, 1167.

ELLIOTT, L. 1968 Two-dimensional boundary-layer theory with strong blowing. *Quart. J. Mech. Appl. Math.* **21**, 77.

EMMONS, H. W. & LEIGH, D. C. 1954 Tabulation of the Blasius function with blowing and suction. *Aero. Res. Counc. Current Paper*, no. 157.

KASSOY, D. R. 1970 On laminar boundary layer blowoff. *SIAM J. Appl. Math.* **18**, 29.

KASSOY, D. R. 1971 On laminar boundary layer blowoff. Part 2. *J. Fluid Mech.* **48**, 209.

LEES, L. & CHAPKIS, R. L. 1969 Surface mass injection at supersonic and hypersonic speeds as a problem in turbulent mixing: Part 1. Two-dimensional flow. *A.I.A.A. J.* **7**, 671.

LEW, H. G. & FANUCCI, J. B. 1955 On the laminar compressible boundary layer over a flat plate with suction or injection. *J. Aero. Sci.* **22**, 589.

LOCK, R. C. 1951 The velocity distribution in the laminar boundary layer between parallel streams. *Quart. J. Mech. Appl. Math.* **4**, 42.

PRETSCH, J. 1944 Analytic solutions of the laminar boundary layer with asymptotic suction and injection. *Z. angew. Math. Mech.* **24**, 264.

ROSENHEAD, L. (ed.) 1963 *Laminar Boundary Layers*. Oxford University Press.

THOMAS, P. D. 1969 Flow over a finite plate with massive injection. *A.I.A.A. J.* **7**, 681.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.

WALLACE, J. & KEMP, N. 1969 Similarity solutions to the massive blowing problem. *A.I.A.A. J.* **7**, 1517.

WATSON, E. J. 1966 The equation of similar profiles in boundary layer theory with strong blowing. *Proc. Roy. Soc.* A **294**, 208.